SIMILARITY PROBLEMS OF THE DYNAMIC AXISYMMETRIC BENDING OF NONLINEARLY ELASTIC PLATES

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UDC 539.3:534.1

A large number of studies have been devoted to problems concerning the deformation of nonlinearly elastic bodies. The mathematical complexity of the topics studied in these investigations accounts for the wide use of approximate methods of solution or of numerical techniques employing a computer. Similarity solutions occupy a special place in a number of these problems. Such solutions can be obtained if certain assumptions are made in the initial conditions of the problem. However, the subsequent steps to obtaining the solution require that the corresponding operations be performed with a high degree of accuracy. It is sometimes possible to obtain a solution in quadratures or in the general case to reduce the problem to the integration of ordinary differential equations. Algorithms for solving the latter on a computer are well-established. From this vantage point, similarity solutions can be considered exact.

Here, we examine the axisymmetric bending of a nonlinearly elastic plate subjected to an unsteady dynamic load. We use a power law to describe the relation between the stresses and strains.

1. We direct the z axis along the symmetry axis perpendicular to the middle plane of the undeformed plate. The position of any point can be determined by the cylindrical coordinates r, θ, and z. Here, r is the length of the position vector located in the middle plane of the undeformed plate and having its origin on the symmetry axis; θ is the angle determining the direction of the vector; z is the distance of the point above the end of the position vector. In the case of axisymmetric deformation, the bending elements are independent of the angle θ.

We will use σ_1 and σ_2 to denote the normal stresses acting over annular and radial sections of the plate. The third stress component acts along a normal to sections parallel to the middle surface. This component is commensurate with the pressure from the transverse load and is not considered in the analysis of the stress state. Some of the shear stresses in the radial and annular sections are equal to zero, by virtue of the symmetry of the strains. The effect of shear stresses, caused by tangential forces, on the deformation of the plate will also be ignored.

The stresses in an elastic isotropic body are represented as the sum of the components of the tensors of the mean stress and stress deviator [1-3]. The first tensor depends on the spherical strain tensor, while the second depends on the deviator of the strain tensor. The relationship between the stresses and strains which follows from this connection can be written as follows for the case being examined:

$$\sigma_j - \sigma = 2G(\varepsilon_j - \varepsilon) \quad (j = 1,\ 2); \tag{1.1}$$

$$\sigma = 3K\varepsilon. \tag{1.2}$$

Here, σ is the mean stress: $\sigma = (\sigma_1 + \sigma_2)/3$; ε is the mean extension: $\varepsilon = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)/3$; ε_1, ε_2, ε_3 are the extensions in the direction of the normals to the annular and radial sections and the middle surface of the plate; K is the generalized tensile bulk modulus, dependent on ε; G is the generalized shear modulus, dependent on the strain intensity ε_*: $G = \sigma_*/(3\varepsilon_*)$; σ_* is the stress intensity:

$$\sigma_* = (\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2)^{1/2},$$
$$\varepsilon_* = (\sqrt{2}/3)\,[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2]^{1/2}. \tag{1.3}$$

Leningrad. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 128-135, September-October, 1989. Original article submitted July 9, 1987.

We will assume that a power relationship exists between the stress intensity and strain intensity

$$\sigma_* = E\varepsilon_*^{\mu} \quad \text{or} \quad G = (E/3)\,\varepsilon_*^{\mu-1} \tag{1.4}$$

(E and μ are an assigned constant and exponent). We rewrite Eqs. (1.1) after substituting (1.2) into them:

$$\sigma_j - \sigma[1 - 2G/(3K)] = 2G\varepsilon_j \quad (j = 1,\ 2). \tag{1.5}$$

In a similarity solution, the stresses and strains must have the following structure: $\sigma_j = t^x\Phi_j(\eta)$, $\varepsilon_j = t^y e_j(\eta)$ $(j = 1, 2)$, where t is time; $\Phi_j(\eta)$, $e_j(\eta)$ are functions of the dimensionless variable η: $\eta = r/(bt^{\beta})$, b is a dimensional constant; x, y, and β are exponents chosen during the course of the solution. When we insert these expressions into (1.5), we obtain equations linking the functions $\Phi_j(\eta)$ and $e_j(\eta)$. The time need not be present explicitly in the equations, due to assignment of exponential relation (1.4) and variation of the values of x and y. However, it may be difficult to exclude the time from the combination G/K in (1.5), thus disturbing the similarity of the solution. The solution will for certain be similar if the problem is linear (G/K = const) or if the material is assumed to be incompressible $(K \to \infty)$.

We will study the last case. It follows from (1.1) that for an incompressible material $(\varepsilon = 0)$

$$\sigma_1 = 4G(\varepsilon_1 + 0.5\varepsilon_2),\ \sigma_2 = 4G(\varepsilon_2 + 0.5\varepsilon_1). \tag{1.6}$$

The hypothesis of straight normals is used to examine bending. A positive value of deflection w corresponds to deviation in the positive direction of the z axis. In the case of axisymmetric deformation, the extensions in the radial and annular directions are determined by the expressions [1]

$$\varepsilon_1 = -z_*\varkappa_1, \quad \varepsilon_2 = -z_*\varkappa_2, \quad \varkappa_1 = \partial^2 w/\partial r^2, \quad \varkappa_2 = r^{-1}\partial w/\partial r \tag{1.7}$$

(z_* is the distance from the point in question to the middle surface of the plate; κ_1 and κ_2 are the curvatures of the deformed surface of the plate in the radial and annular directions). We can use the first two formulas of (1.7) to express (1.3) and (1.4) through κ_1 and κ_2 for any point of the plate:

$$\varepsilon_* = (4H/3)^{1/2}\,|z_*|, \quad G = (E/3)(4H/3)^{(\mu-1)/2}\,|z_*|^{\mu-1}$$

$(H = \kappa_1^2 + \kappa_2^2 + \kappa_1\kappa_2)$. The stresses (1.6) are also expressed through κ_1, κ_2 and are inserted into the formulas for the bending moments $M_j = -\int_{-h/2}^{h/2} \sigma_j z_* dz_*$ $(j = 1, 2)$ (h is the thickness of the plate). We finally obtain

$$M_1 = DH^{(\mu-1)/2}(\varkappa_1 + 0.5\varkappa_2),\ M_2 = DH^{(\mu-1)/2}(\varkappa_2 + 0.5\varkappa_1), \tag{1.8}$$

where M_1 and M_2 are the bending moments per unit width of the annular and radial sections; $D = Eh^{\mu+2}3^{-(1+\mu)/2}(\mu + 2)^{-1}$. We use the known moments (1.8) to find the bending stresses

$$\sigma_j = -M_j\,|z_*|^{\mu}\,h^{-(\mu+2)}2^{1+\mu}(\mu + 2)\,\mathrm{sgn}\,z_* \quad (j = 1,\ 2).$$

We use the conditions of dynamic equilibrium of an element of the plate for axisymmetric bending to obtain

$$\partial M_1/\partial r + (M_1 - M_2)/r = Q,\ \partial Q/\partial r + Q/r = q - m\partial^2 w/\partial t^2. \tag{1.9}$$

Here, Q is the shearing force per unit width of the annular section; q is the transverse load (assumed to be positive when acting in the direction of the z axis); m is the mass of a unit area of the plate. Equations (1.8-1.9), together with the last two equations of (1.7), solve the stated problem.

We seek the similarity solution in the form [4, 5]

$$w = w_* n_1 t^\alpha \varphi(\xi), \quad Q = Q_* n_4 t^\gamma \chi(\xi),$$
$$M_1 = M_* n_3 t^\delta \psi_1(\xi), \quad M_2 = M_* n_3 t^\delta \psi_2(\xi), \quad \xi = r/(n_2 b t^\beta), \tag{1.10}$$

where w_*, Q_*, M_*, b are dimensional constants; n_i (i = 1, 2, 3, 4), α, β, γ, δ are as-yet-unknown numerical coefficients and an exponent; $\varphi(\xi)$, $\psi_1(\xi)$, $\psi_2(\xi)$, $\chi(\xi)$ are sought dimensionless functions of the variable ξ. The load

$$q = q_* t^\omega f(\xi) \tag{1.11}$$

[q_* is a dimensional constant; ω is an assigned exponent; f(ξ) is an assigned dimensionless function of ξ]. Using dimensional analysis, we express all of the dimensional constants through q_*, m, and D:

$$w_* = q_*/m, \quad b = (D q_*^{\mu-1} m^{-\mu})^{1/(2\mu+2)},$$
$$M_* = (D q_*^{2\mu} m^{-\mu})^{1/(\mu+1)}, \quad Q_* = (D q_*^{1+3\mu} m^{-\mu})^{1/(2\mu+2)}. \tag{1.12}$$

The variables of (1.10) and (1.11) are inserted into Eqs. (1.8) and (1.9) and the last two equations of (1.7). In this case, Eqs. (1.12) are taken into account. The time t appears explicitly during the substitutions. To exclude it, the exponent with t is equated to zero. This gives us the relations $(\alpha - 2\beta)\mu - \delta = 0$, $\delta - \gamma - \beta = 0$, $\gamma - \beta - \alpha + 2 = 0$, $\omega - \alpha + 2 = 0$ which we use to find the values of the exponents in (1.10):

$$\alpha = \omega + 2, \quad \beta = \frac{2\mu + \omega(\mu - 1)}{2(1+\mu)}, \quad \gamma = \frac{2\mu + \omega(3\mu + 1)}{2(1+\mu)}, \quad \delta = 2\mu \frac{\omega + 1}{1 + \mu}. \tag{1.13}$$

We finally arrive at the system of equations

$$\psi_1 = n_3^{-1}(n_1 n_2^{-2})^\mu (\xi\varphi'' + 0.5\varphi')\xi^{-\mu} L^{\mu-1},$$
$$\psi_2 = n_3^{-1}(n_1 n_2^{-2})^\mu (\varphi' + 0.5\xi\varphi'')\xi^{-\mu} L^{\mu-1}, \tag{1.14}$$
$$n_3(n_4 n_2)^{-1}[\psi_1' + (\psi_1 - \psi_2)/\xi] = \chi,$$
$$n_4 n_2^{-1}(\chi' + \chi/\xi) = f(\xi) - n_1[\beta^2\xi^2\varphi'' - \beta(2\alpha - \beta - 1)\xi\varphi' + \alpha(\alpha - 1)\varphi]$$

($L = [(\xi\varphi'')^2 + \xi\varphi'\varphi'' + (\varphi')^2]^{1/2}$, with the primes denoting differentiation with respect to ξ).

2. Using the equations obtained above, we will examine the bending of an infinite plate subjected to a load applied in a certain region of the plate. It is known [6] that in the case of the action of a concentrated force, the bending moment and shearing force in the neighborhood of its application vanish. The reason for this is the imprecision of the engineering theory of bending, which does not consider local strains of the material under a force. The exact solutions of the theory of elasticity give infinitely large stresses only directly at the point of load application. The presence of these stresses also cannot always be considered physically correct. Thus, the force is often replaced by a load distributed over a small but finite area [6], which leads to a finite value for the stresses. This is also a natural approach to take in the solution of a dynamic problem involving the local action of a force on a plate. However, fixing the size of the area over which the load acts introduces an additional linear dimension, which disturbs the similarity of the problem. To maintain similitude, it is necessary to assume that q is a function of the variable ξ (1.11). In this case, the load is not fixed relative to the coordinates r of the plate but instead moves over its surface with time. We take the load in the form (1.11), having supposed that

$$f(\xi) = \exp(-k\xi^2). \tag{2.1}$$

By changing k, we can change the area covered by the main part of the load by a certain moment of time. In this case, the load is more or less concentrated. The total force created by the load (1.11), (2.1),

$$S = \int_0^{2\pi}\int_0^\infty qr\,dr\,d\theta = S_* t^\lambda \quad (S_* = \pi q_*(n_2 b)^2 k^{-1}, \ \lambda = (\omega + 1)2\mu/(\mu + 1)). \tag{2.2}$$

We can assign the load on the plate by using (2.2) instead of q_* and ω. The coefficients of (1.13) are expressed through λ (2.2):

$$\alpha = 1 + \lambda(\mu + 1)(2\mu)^{-1}, \quad \beta = 0{,}5 + \lambda(\mu - 1)(4\mu)^{-1}, \quad \gamma = -0.5 + \lambda(3\mu + 1)(4\mu)^{-1}, \ \delta = \lambda. \tag{2.3}$$

These formulas show that at $\lambda \geq 0$ the moments M_1 and M_2 do not vanish at infinity ($\delta \geq 0$). The coefficient γ may have a negative value, i.e., the shearing force Q (1.10) in the neighborhood of the symmetry axis of the plate may be very large at $t \to 0$. This is due to the fact that at the initial moment of time ($t \to 0$), the load distribution on the plate determined by (1.11) and (2.1) is in the nature of an impulse function [7] ($\omega < 0$).

Let us examine the case when the total force does not change with time ($\lambda = 0$). It follows from (2.3) and (2.2) that

$$\alpha = 1, \ \beta = 0.5, \ \gamma = -0.5, \ \delta = 0, \ \omega = -1. \tag{2.4}$$

The coefficient μ drops out of Eqs. (2.4). This means that the exponents with t in (1.10) will be identical, regardless of the elastic properties of the material. In particular, the bending moment at the center of the plate will remain unchanged ($\delta = 0$). Inserting α and β from (2.4) into (1.14), we choose n_i ($i = 1, 2, 3, 4$) such that the form of the equations is simplified: $n_1 = 4$, $n_2 = 2^{\mu/(\mu+1)}$, $n_3 = 2^{2\mu/(\mu+1)}$, $n_4 = 2^{\mu/(\mu+1)}$. As a result, we find from (1.14) that

$$\psi_1 = (\xi\varphi'' + 0.5\varphi')\xi^{-\mu}L^{\mu-1}, \ \psi_2 = (\varphi' + 0.5\xi\varphi'')\xi^{-\mu}L^{\mu-1},$$
$$\psi_1' + (\psi_1 - \psi_2)/\xi = \chi, \quad \chi' + \chi/\xi = f(\xi) - \xi^2\varphi'' + \xi\varphi', \tag{2.5}$$

where $f(\xi)$ is given by Eq. (2.1). The first two equations of (2.5) are rewritten in the form

$$\varphi'' = [(\varphi'')^2 + \varphi''A + A^2]^{(1-\mu)/2}\psi_1 - 0.5A,$$
$$\psi_2 = \psi_1(2A + \varphi'')/(2\varphi'' + A) \ (A = \varphi'/\xi). \tag{2.6}$$

Let us examine the boundary conditions. Due to the symmetry of the bending relative to the center of the plate, the angle of rotation at $r = 0$ is equal to zero and $Q = 0$, while $M_1 = M_2$. On the basis of (1.10) we obtain

$$\xi = 0\colon \ \varphi'(0) = 0, \ \chi(0) = 0, \ \psi_1(0) = \psi_2(0). \tag{2.7}$$

For an infinite plate at $\xi \to \infty$, all of the bending elements tend toward zero. The initial conditions are zero conditions. Here, we must augment these conditions by expressions which develop the indeterminacy of the terms entering into (2.5) and (2.6) at $\xi \to 0$. It is obvious that $\lim_{\xi\to 0} \chi/\xi = \chi'(0)$, $\lim_{\xi\to 0} A = \lim_{\xi\to 0} \varphi'/\xi = \varphi''(0)$. Using these relations in the last equation of (2.5) and the first equation of (2.6), at $\xi = 0$, we find

$$\lim_{\xi\to 0} \chi/\xi = 0.5f(0), \quad A(0) = |3^{-(1+\mu)/2} 2\psi_1(0)|^{1/\mu} \operatorname{sgn} \psi_1(0). \tag{2.8}$$

Due to symmetry considerations, $\lim_{\xi\to 0} (\psi_1 - \psi_2)/\xi = 0$.

The equations can be integrated by the method of reduction to a Cauchy problem. To do this, along with the use of conditions (2.7), we need at $\xi = 0$ to assign $\psi_1(0)$ in (2.8). We also need to assign $\varphi(0)$. Then we integrate. The values of $\psi_1(0)$ and $\varphi(0)$ that are initially adopted need to be corrected in accordance with the degree of nonclosure of the solution at $\xi \to \infty$. We then repeat the process. Variation of the two quantities $\psi_1(0)$ and $\varphi(0)$ leads to a large amount of calculation. However, for the investigated value $\lambda = 0$, the system of equations does not explicitly contain the functions $\varphi(\xi)$. This reduces the order of the system and makes it possible to use only one condition $\psi_1(0)$ to perform the trial runs. After this is done, to find $\varphi(\xi)$ it is sufficient to integrate $d\varphi/d\xi = \varphi'(\xi)$ (here, the right side has already been found). This is a linear problem which requires a single run to find $\varphi(0)$.

Both integrations are actually combined in a single program and are performed at the same time. We used a subroutine based on the Runge–Kutta method for the integration. The program provided for automatic selection of the integration step and involved the solution of a system of equations in higher derivatives. Nevertheless, the first equation of (2.6) is not solved analytically in φ'' with an arbitrary μ. In the course of the calculations, φ'' was determined from (2.6) by the method of successive approximations. As the first approximation, we took the value obtained from the previous step. The process of successive approximation turned out to be convergent in all of the computations performed on the SM-4 computer. Figure 1 shows graphs of the quantities characterizing the bending of the plate for $\mu = 1/3$ and $k = 1$.

It is useful to compare the solution we obtained here with the solution for a linearly elastic plate ($\mu = 1$). The bending of linearly elastic plates subjected to dynamic loading by concentrated forces was studied in [8-10]. We will find the solution for the load (1.11). A similarity solution can be found for a linearly elastic plate by assuming that the plate material is compressible. In this case, the realtion between the bending moments and curvatures has the form [11]

$$M_1 = D_*(\varkappa_1 + \nu\varkappa_2),\quad M_2 = D_*(\varkappa_2 + \nu\varkappa_1),$$

where ν is the Poisson's ratio; D_* is the cylindrical stiffness of the linearly elastic plate; $D_* = E_*h^3/[12(1 - \nu^2)]$; E_* is the modulus of normal elasticity. It follows from these equalities that

$$\psi_1 = \varphi'' + \nu\varphi'/\xi,\ \psi_2 = \varphi'/\xi + \nu\varphi''. \tag{2.9}$$

In (1.12) we replaced D by D_* for $\mu = 1$.

We will again examine the case in which the total force S (2.2) remains constant ($\lambda = 0$). The solution reduces to integration of system (2.5), where the first two equations are replaced by (2.9). This system, written for $\lambda = 0$, makes it possible to introduce the following new variable in the linear case

$$\Phi = \varphi'' - \varphi'/\xi. \tag{2.10}$$

We find from (2.9) and the last two equations of (2.5) that

$$\psi_1 = \Phi + (1 + \nu)\varphi'/\xi,\ \psi_2 = -\Phi + (1 + \nu)\varphi'',\quad \psi_1 - \psi_2 = (1 - \nu)\Phi,\ \chi = \Phi' + 2\Phi/\xi; \tag{2.11}$$

$$\Phi'' + 3\Phi'/\xi + \xi^2\Phi = f(\xi). \tag{2.12}$$

Taking (2.7) into account and evaluating the indeterminate forms at $\xi \to 0$, we use the third and fourth equations of (2.11) to find $\Phi(0) = 0$, $\Phi'(0) = 0$. Also, on the basis of (2.12), $\lim_{\xi\to 0} \Phi'/\xi = \Phi''(0) = 0.25f(0)$. These conditions are sufficient for the integration of (2.12). The function Φ is independent of the a priori unknown boundary conditions $\varphi(0)$, $\psi_1(0)$ and is obtained by means of a single integration. To determine the other quantities, we integrate (2.10) with the found $\Phi(\xi)$ and we use Eqs. (2.11). The indeterminate form entering into (2.10) is evaluated at $\xi = 0$ by means of the first expression of (2.11):

$$\lim_{\xi\to 0} \varphi'/\xi = (1 + \nu)^{-1}\psi_1(0). \tag{2.13}$$

The values of $\psi_1(0)$ and $\varphi(0)$ should be assigned during the trial runs conducted to integrate (2.10). As in the nonlinear case, the calculations are made easier by the fact that (2.10) contains only derivatives of φ. Thus, it is sufficient to vary the one boundary condition $\psi_1(0)$. In determining the function φ we finally refine the value of $\varphi(0)$. Nearly all of the integration operations are performed simultaneously in the computing program. It follows from (2.12) that the function Φ is independent of ν. It goes only into Eq. (2.13), which is used to integrate (2.10). This makes it possible, by means of a simple conversion, to find results for different ν (with $\nu = 0.3$ and 0.5, the difference in the deflections and moments for $\xi = 0$ is about 7-11%).

Figure 2 shows graphs of the quantities calculated for the linearly elastic plate with $k = 1$. They show that in both the linear and nonlinear cases (see Fig. 1), ψ_2 is greater in absolute value than ψ_1 in the region of the plate adjacent to the symmetry axis ($\xi = 0$) i.e., $|M_2| \geq |M_1|$ in this part of the plate. While propagating, the wave associated with the bending moments and shearing force form a precursor which moves out in advance of the expansion of the region of significant plate deflections.

Quantitative comparison of the quantities (1.10) for the linear and nonlinear cases is made difficult by the fact that the coefficients (1.12) entering into these quantities will be different for materials with different elastic properties. Thus, the coefficients are reduced to another form. They are expressed through the quantities determined for a linear

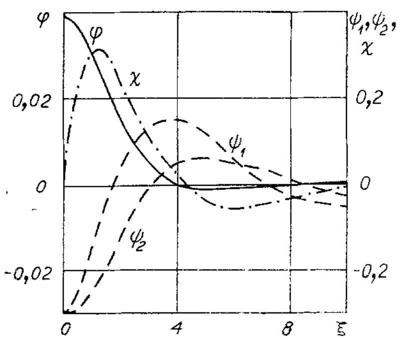

Fig. 1

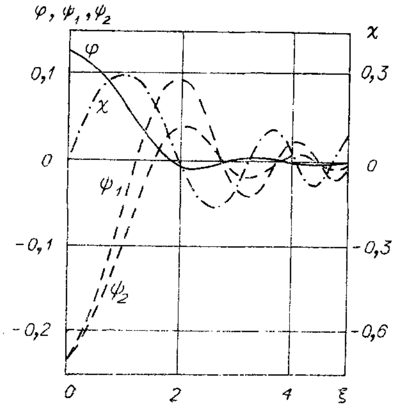

Fig. 2

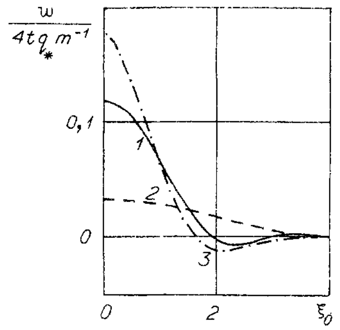

Fig. 3

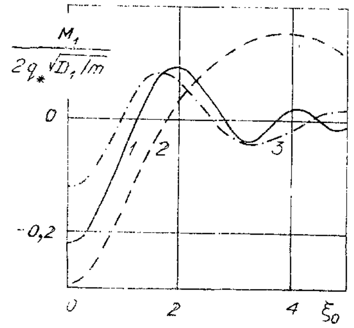

Fig. 4

plate and a certain parameter a, which is dependent on the ratio of the characteristics in the linear and nonlinear cases:

$$b = (D_1/m)^{1/4} a^{-1/2}, \quad M_* = q_* (D_1/m)^{1/2} a^{-1}, \quad Q_* = q_* (D_1/m)^{1/4} a^{-1/2},$$

where $a = D_1^{1/2} D^{-1/(\mu+1)} (q_*^2/m)^{(1-\mu)/(2\mu+2)} = (E_1/3)^{1/2}[(\mu + 2)/E]^{1/(1+\mu)}(q_*^2 m^{-1} \times h^{-1})^{(1-\mu)/(2\mu+2)}$; $D_1 = E_1 h^3/9$; E_1 is the coefficient in Eq. (1.4) for the linear case $\mu = 1$ [it differs somewhat from the elastic modulus E_* determined with uniaxial tension: $E_1 = 3(2 + 2\nu)^{-1} E_*$]. In subsequently comparing the linear and nonlinear problems, the material is assumed to be imcompressible ($\nu = 0.5$). In order to compare the solutions, they must have been calculated for the same loads (1.11), (2.1). Taking into account that the variable ξ (1.10) entering into (2.1) is connected with the elastic properties of the material, when we cacluate the nonlinear plate we need to take a value of the coefficient k which is $(2^{(1-\mu)/(1+\mu)} a)$ times less than in the linear case.

Figures 3 and 4 show graphs of $w/(4tq_* m^{-1})$ and $M_1/(2q_* \sqrt{D_1/m})$ characterizing the deflection and bending moment with $\lambda = 0$. The quantity $\xi_0 = r/(4t^2 D_1/m)^{1/4}$ is plotted off the x-axis. Line 1 corresponds to the linear case calculated with k = 1 (2.1). Lines 2 and 3 were constructed for a nonlinear plate at $\mu = 1/3$ for $a = 0.707$ and 2.83. These parameters correspond to the values k = 1 and 0.25. At these values, the load (2.1) remains equivalent for the linear and nonlinear plates. In the case of large a (line 3), the deflection is more localized near the symmetry axis and increases more rapidly than in the linear case. Here, the moment turns out to be smaller. The opposite pattern is seen for small a (line 2). It can be assumed that a characterizes the compliance of the plate and, in turn, depends on the ratio of E_1 and E.

Figure 5 shows graphs (lines 1) illustrating the dependence of (1.4) on E with fixed μ ($\mu < 1$). Also shown is the straight line for a linearly elastic material at a certain E_1. An increase in E is accompanied by a shifting of the curves upward, thereby increasing the range of strains for which the nonlinear material turns out to be stiffer than the linear material. This explains the results obtained from comparing the solutions.

3. It is very interesting to study the bending of plates of limited size. However, the solution can be similar only in the special case when the exponent β in ξ (1.10) is

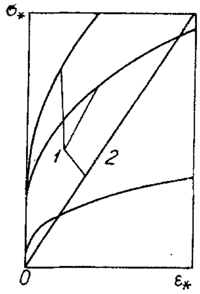

Fig. 5

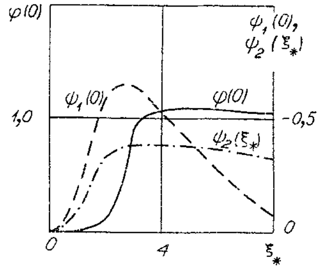

Fig. 6

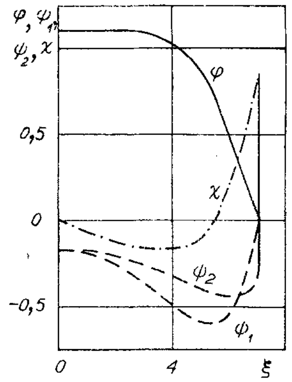

Fig. 7

equal to zero. Then ξ will depend only on the space coordinate r, which makes it possible to satisfy boundary conditions assigned a finite distance from the center. We will refer to the corresponding solution as degenerate [12]. For this, from (1.13) with $\beta = 0$, $\alpha = 2/(1 - \mu)$, $\gamma = \delta = \omega = 2\mu/(1 - \mu)$. Here, the value of μ determines the value of ω, which in turn prescribes the character of change in the load over time (1.11). For n_i (i = 1, 2, 3, 4) in (1.14), it is best to take values different from those taken previously:

$$n_1 = [\alpha(\alpha - 1)]^{-1}, n_2 = n_4 = [\alpha(\alpha - 1)]^{-\mu/(2\mu+2)}, n_3 = [\alpha(\alpha - 1)]^{-\mu/(\mu+1)}.$$

In this case, the first three equations of (2.5) and Eq. (2.6) remain valid. Instead of the last equation we obtain

$$\chi' + \chi/\xi = f(\xi) - \varphi. \tag{3.1}$$

The second condition (2.8) is kept, while the first is replaced by $\lim_{\xi\to 0} \chi/\xi = \chi'(0) = 0.5[f(0) - \varphi(0)]$. The boundary conditions at the center of the plate (2.7) remain as before.

Let us examine the action of a uniformly distributed load on a plate simply supported at r = R by an annular support. We have $\xi = \xi_*: \varphi(\xi_*) = 0$, $\psi_1(\xi_*) = 0$, where $\xi_* = R/(n_2 b)$, and in (3.1) $f(\xi) = 1$. In integrating, we use Eqs. (2.6) and (3.1) and the third equation of (2.5). The integration process is similar to that described in Part 2. However, now we need to simultaneously vary two a priori unknown conditions $\varphi(0)$ and $\psi_1(0)$ at $\xi = 0$ in order to satisfy the boundary condition with $\xi = \xi_*$. Although this entails a large amount of calculation, simplifications can be made. Let the order of the quantities $\varphi(0)$ and $\psi_1(0)$ be known. We fix the value of one of them, say $\varphi(0)$, and we solve a series of Cauchy problems. Here, we change $\psi_1(0)$ in each solution. In solving the series of problems, we determine the values of ξ_1 and ξ_2 at which $\varphi(\xi_1)$ and $\psi_1(\xi_2)$ vanish. The goal of the process is to find the value of $\psi_1(0)$ for which ξ_1 coincides with ξ_2. The value $\xi_1 = \xi_2$ can be regarded as the support coordinate of the plate ξ_*, where the boundary conditions $\varphi(\xi_*) = 0$, $\psi_1(\xi_*) = 0$ are satisfied simultaneously. In the given procedure, the $\varphi(0)$, sought quantity is ξ_*, i.e., the radius of the plate R. Having taken another value of $\varphi(0)$, we can calculate the strain of the plate for another support radius. A series of such calculations performed for $\mu = 1/3$ ($\omega = 1$) allowed us to construct (in Fig. 6) graphs of the dependences of $\varphi(0)$, $\psi_1(0)$, $\psi_2(\xi_*)$ on ξ_*. These dependences characterize the deflection and the bending moments M_1 at the center of the plate and M_2 at its support as a function of the size of the plate.

Dynamically applied loads are balanced by elastic and inertial forces. Given small plate dimensions ($\xi_* < 2$), elastic forces predominate. At $\xi_* \to \infty$, the plate resists only with inertial forces, and $\varphi(0) \to 1$. For short plates ($\xi_* < 3$), the maximum values of M_1 and M_2 are seen at the center. Here, M_2 changes slightly (up to 10%) along the radial section to the value r = 0.6R. Only at the support does it turn out to be 40% less than at the center of the plate. With an increase in the radius of the plate, strains begin to be localized next to the supports. Thus, the maxima of M_1 and M_2 are displaced from the plate's center toward the support. Figure 7 shows the distribution along the plate radius of quantities characterizing bending elements. Here $\xi_* = 7.05$.

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MECHANICAL MODEL OF AN ELASTOPLASTIC BODY

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UDC 539.3

It is presently held that the phenomenon of the plastic deformation of solids is based on the shear or slip of one part of the material over another [1-12]. Despite the agreement on the nature of plastic deformation, different approaches have been taken to describe the phenomenon. One school of thought is that plastic deformation is governed by a shearing process which takes place in a whole fan of slip planes [1-3]. Other investigators [4-12] believe that such a process occurs only in a finite system of slip planes with a particular orientation. In [4-8], this system was associated with the set of planes acted upon by the principle shear stresses. Another set was hypothesized to be composed of equally-inclined or octahedral planes [9, 10]. As regards macroscopic studies, they do not contradict any of the approaches taken [3, 5, 13, 14], but they do show that the last-mentioned methods have certain advantages: the beginning of plastic deformation is described best by the condition of constancy of the octahedral shear stress (or von Mises condition) [12, 15]; during simple loading, the "single" curve hypothesis, establishing the dependence of the octahedral shear strain on the octahedral shear stress [16-18], turns out to be valid. Microscopic studies undertaken to determine the planes of slip in a body being deformed also have failed to resolve the problem of selecting an approach. This is because actual materials are to a known

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 136-144, September-October, 1989. Original article submitted March 11, 1988; revision submitted April 21, 1988.